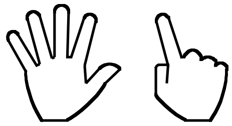
Visualizing Mathematics

Joan A. Cotter, Ph.D.



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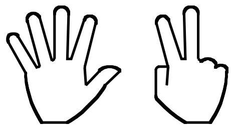


7

Yellow is the Sun

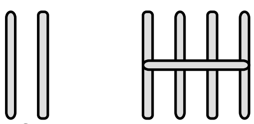
Yellow is the sun.
Six is five and one.

Why is the sky so blue?
Seven is five and two.

Salty is the sea.
Eight is five and three.

Hear the thunder roar.
Nine is five and four.

Ducks will swim and dive.
Ten is five and five.



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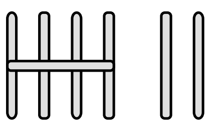

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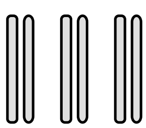
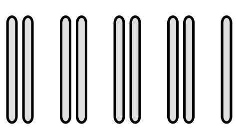
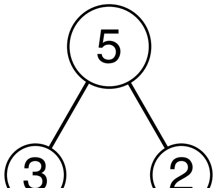


An even number.



An odd number.



Part-part-whole. Helps children solve problems and write equations.

In our concern about the memorization of math facts or solving problems, we must not forget that the root of mathematical study is the creation of mental pictures in the imagination and manipulating those images and relationships using the power of reason and logic.

–Mindy Holte

Economics of mathematics education.

- In international studies, such as TIMSS and PISA, the U.S. scores low compared to other countries.
- In 2004 of the 1.2 million students who took the ACT test, only 40% were deemed ready to study college algebra.
- 25% of college students take remedial math, 37% in CA.
- Only 51% of patents going to U.S. citizens, down from 90%.

Mathematics is changing.

- Our world is becoming increasingly ever more mathematized.
- Math itself is expanding, e.g., fractals, statistics, encryption.
- Computers and calculators both change and are changed by new advances in math.
- Math itself is becoming more visual, e.g., graphing calculators, fractals, encryption. More geometry is being used.

Mathematics education is changing.

- More known about brain and learning, e.g., child under stress stops learning.
- Real learning means making connections.
- Learning styles: majority of children do not learn best by listening. A study showed that teachers spend over 80% time talking.
- Visual thinkers, the gifted and many with LD, find rote memorizing difficult. They need to see whole picture, not small steps.
- Standards suggest what, when, and how math is taught.
- More than arithmetic must be taught: Geometry, algebra, probability, and statistics are all on state exams from K on.

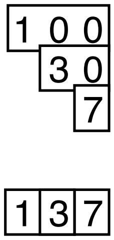
Why understanding is necessary.

- Understanding aids memory: 93 min to learn earn 200 nonsense syllables, 24 min to learn 200 words of prose, and 10 min to learn 200 words of poetry.
- Better learning.
- Less memorization and review needed.
- Essential for applying to real problems.
- Impossible to memorize advanced math.

Counting, a rote activity, does not help child master math concepts.

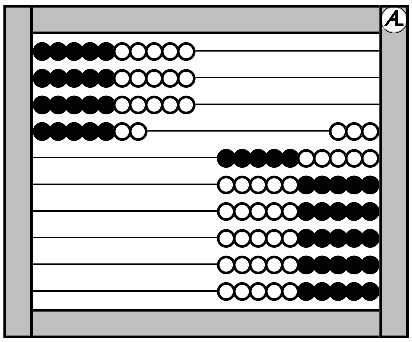
- The alphabet example of how we teach children adding. ($F + E = K$)
- Ignores place value.
- Young children don't realize counting represents quantity.
- Very error prone.
- No motivation to learn facts.
- Inefficient and time-consuming.

Visualizing quantities.
- Babies, at 5 months, can add and subtract up to 3.
- Group by 5s. Impossible to imagine 8 objects without grouping.



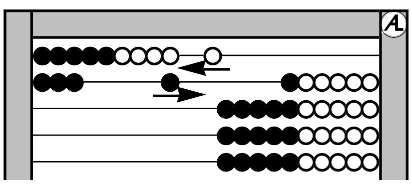
Place value cards.

Place value is the most important concept of arithmetic.
- Teach *math way* of counting: after 10, say ten 1 (11), ten 2 (12), ten 3 (13), . . , 2-ten (20), 2-ten 1 (21), . . . , 9-ten 9 (99).
- All Asian children learn math with math way of naming numbers they understand place value in first grade. Average U.S. child understands it at the end of fourth grade.
- Essential for understanding algorithms.
- Place-value cards: encourage reading in normal order; starting with ones column and then tens columns is backwards.
- Essential to use 4-digit numbers to understand trading (carrying).

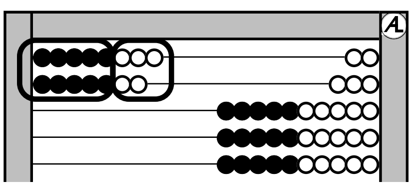

AL abacus (side 1) with 37 entered.

What makes a good manipulative (according to the Japanese)?
- Easily visualized.
- Representative of the structure of mathematics.
- Easily managed.

The AL Abacus.
- Grouped in fives and tens.
- Used for operations, strategies, money.
- Evens and odds; also needed for side 2 of abacus.

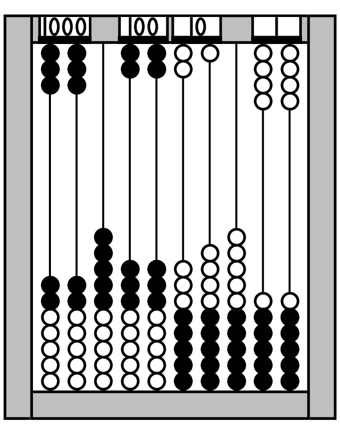

Transforming 9 + 4 into 10 + 3.

Some addition strategies.
- What makes 10: seen on abacus, Go to the Dump game.
- Adding 9: complete the 10.
- Two 5s: two fives = 10; then add “leftovers.” For 8 + 7, the leftovers are 3 + 2; so the sum is 15. See figure at left.


Seeing the sum of 8 and 7 as 10 (2 fives, the black beads) and 5, the number of white beads.

Learning the facts.
- Strategies first: It takes time for new strategy to become automatic.
- Games far superior to flash cards.
- Timed tests and graphs.

Importance of mental computation.
- Understanding more important than procedures.
- Develops number sense (common sense with numbers).
- Necessary for estimating.
- Easier to start at the left: e.g. 34 + 48 = 34 + 40 [74] + 8 [82].



AL abacus (side 2) with 6438 entered.

Adding 4-digit numbers on the abacus.
- Important for understand trading: that 10 ones = 1 ten, 10 tens = 1 hundred, 10 hundreds = 1 thousand.
- Children need to write down on paper what happens after number is added on the abacus.

Introducing subtraction.
- Only after addition is mastered. It is psychologically negative.
- Going up is easier for some facts; e.g. 11 – 9. Used for change.
- 15 – 8: subtract 8 from 10.

Note: This PowerPoint presentation and handout available at www.alabacus.com

Addition Strategies

Strategies are a way to learn a fact or to recall a forgotten fact. These strategies are not to be taught as rules, but should be thought of a powerful visual tool. Counting should be discouraged because it is slow, often inaccurate, and unmindful of place value. Some facts can be learned with more than one strategy. Also teach the names of the strategies.

Number plus 1. Adding 1 to a number is the next number.

Even number plus 2. Adding 2 to an even number is the next even number.

Odd number plus 2. Adding 2 to an odd number is the next odd number.

Adding 5 plus numbers 1 to 4. Adding 5 to a number is obvious on fingers or the abacus.

What makes 10. Enter 10 on the abacus. Separate one quantity and see what's left.

Adding 9. Enter 9 on the first wire and the other number, for example, 4 on the second wire. Take 1 from the 4 and give it to the 9 to make a ten. The sum is 10 plus 3, or 13.

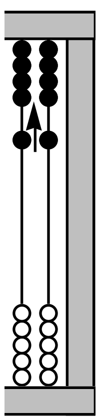
Adding 8. This strategy is similar to adding 9. Two beads are moved back, which is similar to counting by 2s backward.

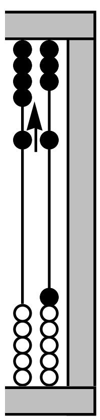
Two-fives. Both numbers need to be between 5 and 10. For example, enter 8 and 7 on two wires of the abacus. The sum is 10 plus the "leftovers," 3 and 2.

Doubles. New facts with the doubles are 3 + 3 and 4 + 4. Beyond 4 + 4 they can be seen as the Two Fives strategy.

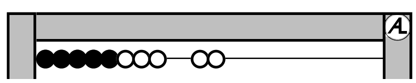
Near Doubles. New facts with the doubles are 3 + 4 and 4 + 5. Beyond 4 + 5 they can be seen as the Two Fives strategy.

Relating facts. There are four facts not covered with these strategies: 6 + 3, 3 + 6, 4 + 7, and 7 + 4. They can be seen in relation to the 10's facts.

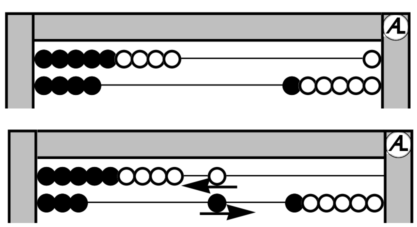




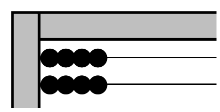
Even + 2. Odd + 2.

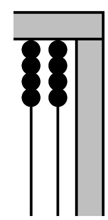


Making 10: 8 + 2 = 10.

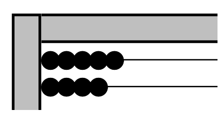


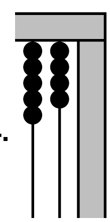
Adding 9: 9 + 4 becomes 10 + 3.



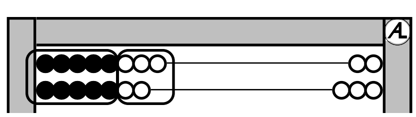


Doubles: 4 + 4.

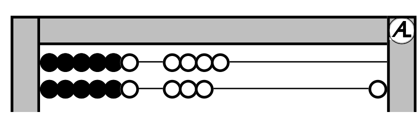




Near doubles: 5 + 4.



Two 5s: 8 + 7 = 10 plus 3 and 2.

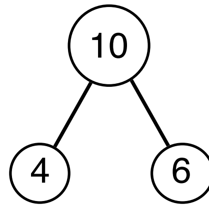


Relating facts. If 6 + 4 = 10, then 6 + 3 = 9.

Subtraction Strategies

A child should know all the addition facts before attempting to master the subtraction facts to avoid confusing the two. Some addition strategies require subtracting and some subtraction strategies require adding. The whole point of strategies is that they be efficient and accurate. Visual strategies are very powerful. Counting is not an efficient strategy.

Some facts can be learned with more than one strategy. Teach the names of the strategies.



Part/whole circles showing 10 = 4 + 6, 10 – 4 = 6, and 10 – 6 = 4.

Part/whole circles. Part/whole circles, as shown below, help children see the correlation between addition and subtraction. The whole is written in the large circle and the parts in the smaller circles. They also help children solve word problems.

Subtracting 1. Subtracting 1 from a number is the previous number.

Subtracting 2. Subtracting 2 from an even number is the previous even number. Subtracting 2 from an odd number is the previous odd number.

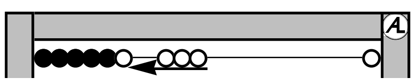
Subtracting consecutive numbers. Explain the meaning of *consecutive*. The result is 1.

Subtracting from 10. These were learned first with the Go to the Dump game.

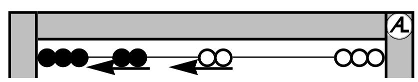
Subtracting from 9 and 11. Do these by comparing them to subtracting from 10.

The facts ≤ 10. The above strategies include all the facts with the following exceptions: 6 – 3, 7 – 3, 7 – 4, 8 – 3, 8 – 4, and 8 – 5. Two 8 facts easily can be seen with 5 as one part. The remaining four facts can be found with the following Going Up strategy.

Going Up < 10. The Going Up strategy works for any fact. To subtract 9 – 6, start with 6 and remember how much is needed to get to 9. [3] If the number being subtracted is less than 5,



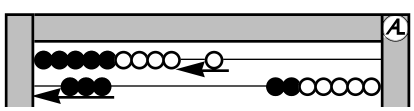
Going up: 9 – 6, start at 6 and go up to 9 by adding 3, the answer.



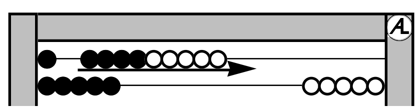
Going up: 7 – 3, start at 3, go up to 5, then go up to 7. The answer is 4, the total added.

first find how much is needed to go to 5 and then add the amount over 5.

Going Up > 10. Use the same procedure for subtracting from numbers over 10. For 13 – 9, it



Going up: 13 – 9, start at 9, go up to 10, then go up to 13. The answer is 4, the total added.



Subtracting from 10: 15 – 9, subtract 9 from the 10, then 1 + 5 = 6.

takes 1 to get to 10 and 3 to get to 13; so the answer is 1 + 3 = 4. See the left figure below.

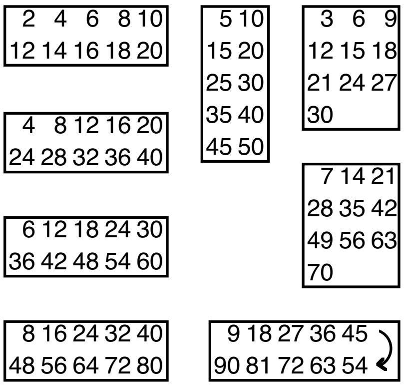
Subtracting from a 10. Subtracting 15 – 9 can be also thought of as subtracting 9 from the 10, getting 1, and adding the result to 5 to get 6. See right figure above.

Derived strategies. Of course, there are also derived strategies. For example, if you know 12 – 6 = 6, then 13 – 6 = 7.

Doubles and near doubles. The doubles and near doubles are not very obvious in subtraction, and therefore, not good strategies.

Multiplication Strategies

Skip counting (multiples).
- Needed for multiplication facts, fractions, and algebra.
- Start as soon as 1-100 is understood; use patterns.

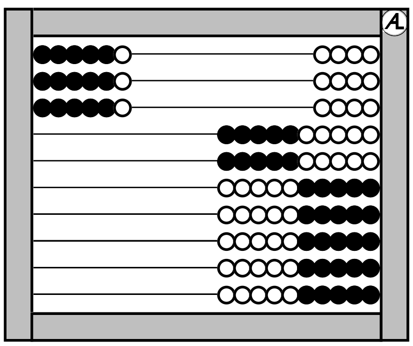


Skip counting patterns.

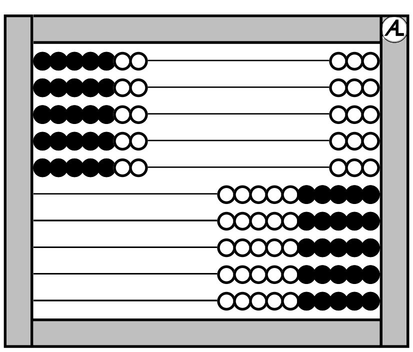
Skip counting pattern explanations.
- Twos. The second row is 10 plus the first row. They are the even numbers.
- Threes. Consider the ones: they increase starting at the lower left with 0 (30) and continue up the first column and over to bottom of the second column and to the third column. Next consider the tens: 0, 1, 2 in each column. Sum of the digits: 3 in the first column (1 + 2, 2 + 1, 3 + 0), 6 in the second column, and 9 in the third.
- Fours. The second row is 20 more than the first row, every other even number.
- Fives. They have an obvious singsong pattern.
- Sixes. The first row is the even 3s. Second row is 30 more than the first row.
- Sevens. Within each row the tens increase by 1. The ones increase by 1 starting at the upper right (21) and continuing down the column and over to the next column.
- Eights. In each row the ones are the decreasing even numbers. The second row is 40 more than the first row, also every other 4.
- Nines. The sum of the digits in all cases is 9. The ones decrease while the tens increase. The second row has the digits of the first row reversed, as shown by the arrow.

Ditties
- Are stored in language, not math, part of the brain.
- A child who learns “6 x 1 = 6, 6 x 2 = 2, . . . “often cannot recognize multiples, which are necessary for simplifying fractions and algebra.

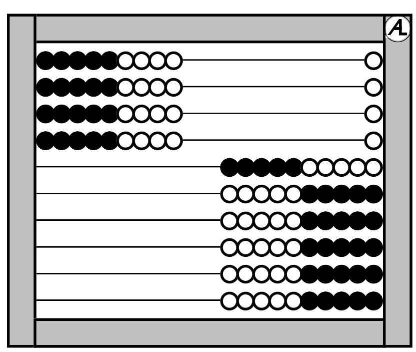
Multiplication seen visually on abacus. (See figures at left and below.)



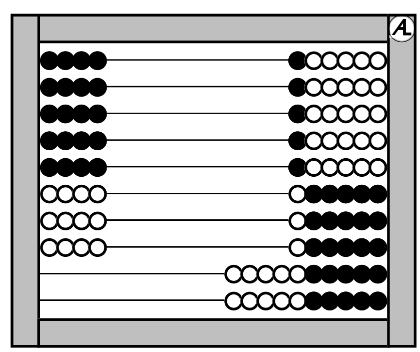
6 x 3 (6 taken 3 times).



7 x 5: see groups of 5s.



9 x 4 = 10 x 4 - 4.



8 x 4: four groups of 5 and 12 more.

SKIP COUNTING MEMORY

Objective To learn the skip counting patterns on page 7.

Preparation To prepare the envelopes, see page 9. The players use the envelopes for reference during the game to memorize the patterns.

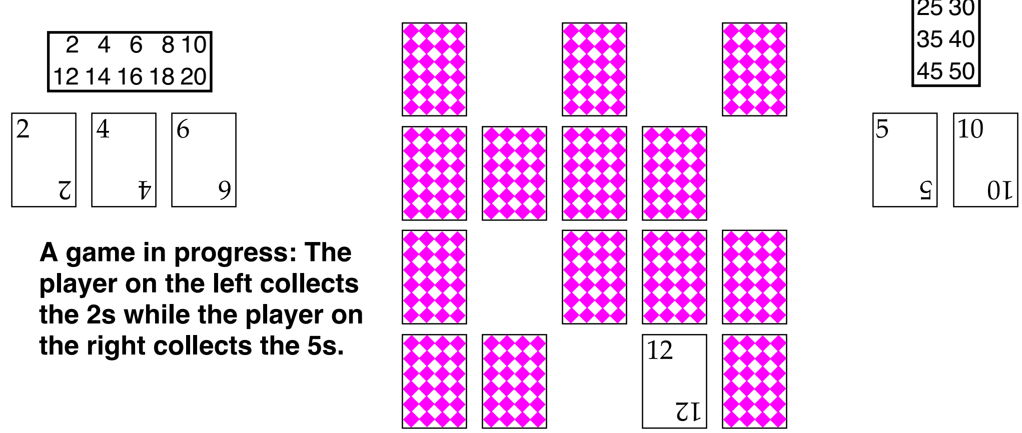
Number of players 2 or 2 teams

Cards Each player chooses an envelope and removes the cards. Mix the cards together and shuffle lightly. Lay the cards out face down in a 5 by 4 array.

Object of the game To be the first player to collect *in order* the complete set of cards

Play The first player turns over *one* card so both players can see it. If it is the needed card, the player collects the card and receives another turn. If it is not the needed card, the card is returned. Next the second player takes a turn. Turns alternate until one player has picked up all ten cards.

Stress the importance of returning the cards to the correct envelopes following a game.



A game in progress: The player on the left collects the 2s while the player on the right collects the 5s.

MULTIPLICATION MEMORY

Objective To help the players master the multiplication facts.

Cards 20 cards: use 10 basic number cards, 1 to 10, and one set of product cards.

Number of players Two. Beginners should sit on the same side of the cards.

Object of the game To collect the most cards by matching the multiplier with the product.

Layout Lay the basic number cards face down in two rows. To the right in separate rows lay the product cards.

Play The first player turns over a basic number card and states the fact. For example, if the card is 4, the player states, "Three taken four times is 12." He then decides where it could be among the product cards. If he is correct, he collects both cards and takes another turn. If he is not, both cards are returned face down in their original place, and the other player takes a turn.